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On Results Between Matrix Division and Some other Matrix Operations

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ABSTRACT: In this study, the relationships between matrix division and some matrix properties such as transpose, simplification, and expansion are discussed. Unless otherwise stated, our matrices are real number matrices. The relationship between our definitions of column co-divisor and row co-divisor of a matrix on another matrix is examined. Equality and differences of these connections are observed. The validity of the provided features is discussed under which conditions. These features have been determined. Examples are given for features that do not. Some of the obtained properties bring different solutions to the solution of linear equation systems, which are encountered in the literature and are more used. This led us to examine the relationship between the defined division operation and transpose. Investigations on this subject enabled the presentation of new definitions, lemmas and theorems. Examination of these offered a broader perspective on the given mathematical expressions.

Keywords: Matrix division, operation of division in matrices, matrix theory, transpose, column co-divisor, row co-divisor.

AMS Subject Classification: 08A40, 15A09, 15A15, 15A80, 65F05, 17C60

1. INTRODUCTION

Here F is a field and $M_n(F)=\left\{\left[a_{ij}\right]_n \middle| a_{ij}\in F, n\in \mathbb{Z}^+\right\}$ is the set of regular matrices. The transpose of $A\in M_n(F)$ is denoted by A^T. The algorithm of row co-divisors is given below, similar to the column co-divisor algorithm. Instead of the F field, the $\mathbb{R}$ real numbers field is taken.

Let $A,B\in M_n(\mathbb{R})$ be any two matrices. The determinant of the new matrix obtained by writing the i^{th} row of the matrix A on the j^{th} row of the matrix B is called the *co-divisor by row* of the matrix A by the row on the matrix B. It is denoted by $\underset{ij}{AB}$. Their number is n^2. The matrix co-divisor by row is $\left[\left(\underset{ij}{AB}\right)_{ij}\right]$ [2,3,4,5,6].

Example 1.1. Let $A=\begin{bmatrix}1&3\\2&5\end{bmatrix}$ and $B=\begin{bmatrix}2&1\\4&7\end{bmatrix}$ be regular matrices. Matrix of co-divisors by row of matrix A on matrix B is $\left[\left(\underset{ij}{AB}\right)_{ij}\right]$.

$$\underset{11}{AB}=\begin{vmatrix}1&3\\4&7\end{vmatrix}=-5,\ \underset{12}{AB}=\begin{vmatrix}2&1\\1&3\end{vmatrix}=5.\ \underset{21}{AB}=\begin{vmatrix}2&5\\4&7\end{vmatrix}=-6,\ \underset{22}{AB}=\begin{vmatrix}2&1\\2&5\end{vmatrix}=8.$$

$$\left[\left(\underset{ij}{AB}\right)_{ij}\right]=\begin{bmatrix}-5&5\\-6&8\end{bmatrix}.$$

Likewise, the matrix of rows co-dividing matrix B over matrix A is matrix $\left[\left(\underset{ij}{BA}\right)_{ij}\right]$.

$$BA_{11}=\begin{vmatrix}2 & 1\\ 2 & 5\end{vmatrix}=8,\ BA_{12}=\begin{vmatrix}1 & 3\\ 2 & 1\end{vmatrix}=-5,\ BA_{21}=\begin{vmatrix}4 & 7\\ 2 & 5\end{vmatrix}=6,\ BA_{22}=\begin{vmatrix}1 & 3\\ 4 & 7\end{vmatrix}=-5.$$

$$\left[\left(BA_{ij}\right)_{ij}\right]=\begin{bmatrix}8 & -5\\ 6 & -5\end{bmatrix}.$$

For the two matrices satisfying the above conditions, the matrix division is also given by $\frac{A}{B}:=\frac{1}{|B|}\left[\left(\frac{A}{B}_{ij}\right)_{ji}\right]$ [2,3,4].

2. THE RESULTS BETWEEN TRANSPOSE AND DIVISION

In the first part of this section, similar properties of the transpose according to the multiplication operation are examined. Secondly, it is investigated whether the transpose provides some of its properties compared to the division operation.

For all $A,B\in M_n(\mathbb{R})$ then,

$$\left(\frac{A}{B}\right)^T\neq\frac{A^T}{B^T}.$$

Example 2.1. Consider the matrices $A=\begin{bmatrix}1 & 3\\ 2 & 5\end{bmatrix}$ and $B=\begin{bmatrix}2 & 1\\ 4 & 7\end{bmatrix}$ given in Example 1.1. above,

$$\left(\frac{A}{B}\right)^T=\begin{bmatrix}\frac{1}{2} & 0\\ \frac{8}{5} & -\frac{1}{5}\end{bmatrix},$$

$$A^T=\begin{bmatrix}1 & 2\\ 3 & 5\end{bmatrix}\text{ and }B^T=\begin{bmatrix}2 & 4\\ 1 & 7\end{bmatrix}$$

$$\frac{A^T}{B^T}=\begin{bmatrix}-\frac{1}{2} & -\frac{3}{5}\\ \frac{1}{2} & \frac{4}{5}\end{bmatrix}.$$

Lemma 2.1. *Let* $A\in M_n(\mathbb{R})$ *be any matrix. Then,*

$$\left(\frac{I_n}{A}\right)^T=\frac{I_n}{A^T}.$$

Proof. Let a regular matrix $A=\left[a_{ij}\right]_n$ be given.

$$\frac{I_n}{A^T}.A^T=I_n\wedge A^T.\frac{I_n}{A^T}=I_n$$

$$\frac{I_n}{A^T}=\left(A^T\right)^{-1}=\left(A^{-1}\right)^T=\left(\frac{I_n}{A}\right)^T$$

$$\frac{I_n}{A^T}=\left(\frac{I}{A}\right)^T.\square$$

The following lemma is given which simply explains the relationship between the row co-divisors matrix and the transpose.

Lemma 2.2. *Let* $A,B\in M_n(\mathbb{R})$. *Then,*

$$\frac{1}{|A|}\left[\left(\underset{ij}{BA}\right)_{ij}\right]=\left(\frac{B^T}{A^T}\right)^T .$$

Proof. For all $A,B\in M_n(\mathbb{F})$ then $\underset{ij}{BA}=\frac{B^T}{A^T}ij$. Because, the row co-divisors of matrix B on matrix A are the same as the column co-divisors of matrix B^T on matrix A^T .

$$\left[\left(\underset{ij}{BA}\right)_{ij}\right]=\left[\left(\frac{B^t}{A^T}ij\right)_{ij}\right]$$

$$\frac{1}{|A|}\left[\left(\underset{ij}{BA}\right)_{ij}\right]=\frac{1}{|A^T|}\left[\left(\frac{B^t}{A^T}ij\right)_{ji}\right]^T=\left(\frac{B^T}{A^T}\right)^T .\square$$

Theorem 2.1. *Let* $A,B\in M_n(\mathbb{F})$ *. Then,*

$$\frac{A}{B}=\frac{B^T}{A^T} .$$

Proof. For all $A,B,X\in M_n(\mathbb{F})$ then $BX=A$. The solution of the linear matrix equation $BX=A$ is $X=\frac{A}{B}$ in [2,4,5]. Then,

$$(BX)^T=A^T \Leftrightarrow X^TB^T=A^T$$

$$X^T=\left(\frac{A^T}{B^T}\right)^T \Leftrightarrow X=\frac{A^T}{B^T}=\frac{A}{B} .\square$$

Theorem 2.2. *Let* $A,B,X\in M_n(\mathbb{F})$ *. Then, solution of the linear matrix equation* $XA=B$ *;*

$$X=\left(\frac{B^T}{A^T}\right)^T .$$

Proof. The solution of the equation $AX=B$ is $X=\frac{B}{A}$, for all $A,B,X\in M_n(F)$. Then

$$(AX)^T=B^T \Leftrightarrow X^TA^T=B^T$$

$$X^T=\frac{1}{|A^T|}\left[\left(\underset{ij}{B^TA^T}\right)ij\right]\Rightarrow X=\frac{1}{|A_T|}\left[\left(\underset{ij}{B^TA^T}\right)_{ji}\right]^T$$

$$X=\frac{1}{|A^T|}\left[\left(\underset{ij}{B^TA^T}\right)_{ij}\right]=\left(\frac{B^T}{A^T}\right)^T .\square$$

Lemma 2.3. *Let* $A\in M_n(\mathbb{F})$ *. Then there are unique regular matrices* A_1 *and* A_2 *such that* $\frac{A}{A^T}=A_1$ *and* $\frac{A}{A^T}=\frac{I_n}{A_2}$.

Proof. Using the factorization of a matrix, the following equations are shown.

$$\frac{A}{A^T}=\frac{A^TA_1}{A^T}=A_1 \text{ and } \frac{A}{A^T}=\frac{A}{AA_2}=\frac{I_n}{A_2} .\square$$

Theorem 2.3. Let $A,B\in M_n(\mathbb{F})$. Then there are unique regular matrices E_1 and E_2 such that $\frac{A}{B}=E_1$ and $\frac{A}{B}=\frac{I_n}{E_2}$.

Proof. Again, considering the factorization of a matrix, the following equations are shown.

$$\frac{A}{B}=\frac{BE_1}{B}=E_1 \text{ and } \frac{A}{B}=\frac{A}{AE_2}=\frac{I_n}{E_2}=E_1 .$$

Corollary. *It is* $E_1E_2 = I_n \wedge E_2E_1 = I_n$ *for* E_1 *and* E_2 *matrices that satisfy these conditions.* □

Theorem 2.4. Let $A \in M_n(\mathbb{R})$. Then,

3. RESULTS

While the solution of the equation $AX = B$ is always sought in the literature, the solution of the equation $XA = B$ is introduced to the literature from now on. There are many matrices corresponding to the $\frac{A}{B}$ rational matrix. The relationship between linear matrix equations and the transpose of a matrix and the division of matrices is determined**.**

4. DISCUSSION

The determination of the relationship between the equations $AX = B$ and $XA = B$ is open to debate. As with real numbers, it is not known whether a rational matrix has the simplest matrix.

5. CONCLUSIONS

With matrix transformations, further studies can be brought to the literature. Transformations, matrix division, and the trap triple are open to study.

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