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Existence Analysis of Conformable Fractional Boundary Value Problems

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ABSTRACT: This study examined the fractional boundary value issue at flexible derivatives, focusing on establishing the existence of solutions and uniqueness. We introduce conditions that advance our understanding of this complex mathematical domain by capitalizing on the innovative framework of contraction principles. Moreover, the versatility of the proposed method is emphasized by its effectiveness in dealing with a wide set for compatible fractional differential equations characterized by diverse boundary conditions.

Keywords: Boundary Value Problem, Fractional differential equations, mapping contractions

1. INTRODUCTION

Since the 17th century, the calculus of fractions, a subfield of mathematics that applies a traditional idea of differentiation and integration to arbitrary orders, has had a long history. Mathematicians such as Leibniz, Euler, and Laplace explored the possibility of fractional derivatives and integrals, but significant progress was not made in this field until the 20th century. The fractional derivative is an extension of the conventional derivative, and several varieties of fractional derivatives have been introduced to date. Some of the earliest definitions of fractional derivatives include Riemann-Liouville Fractional Derivative which was introduced by Riemann and Liouville in the 19th century, is predicated upon the concept of fractional integration and Caputo Fractional Calculus Which was developed by Caputo in the 20th century, that is more suitable for physical applications, as it ensures that fractional derivatives of integer-order functions are also integer-order functions. [1-5].

A novel approach to fractional calculus has recently emerged, gaining significant attention among scientists named conformable fractional derivatives. Introduced by Khalil et al. in 2014 [6, 7], these derivatives propose a more intuitive and straightforward technique for fractional calculus compared to traditional definitions for fractional Calculus like Riemann-Liouville, Caputo Conformable fractional derivatives defined using a limit-based approach, akin to the definition of the classical derivative.

Boundary value problems (BVPs) pertaining to fractional differential equations have emerged as a pivotal area of research, driven by their extensive applications across diverse scientific and engineering domains. The ability of fractional calculus to model complex phenomena with memory and nonlocality has made it an invaluable tool for addressing a wide range of problems in fields such including physics, engineering, economics, and biology. In recent decades, a surge of research has focused on the existence, uniqueness, and positive solutions for various differential equations of fractional order. Scholars have employed various mathematical techniques and tools to investigate these problems, including fixed point theorems, variational methods, and topological degree theory [8-14]. A study of fractional BVPs has resulted in significant progress in our comprehension of complex systems and procedures. By capturing the non-integer order dynamics of these systems, fractional calculus provides valuable insights into phenomena that are difficult to model using traditional integer-order differential equations.

Conformable fractional boundary value problems (CFBVPs), a specialized subset of boundary value problems involving conformable fractional derivatives, have witnessed a surge in interest. While their Riemann-Liouville and Caputo counterparts have been more extensively explored, CFBVPs offer a unique perspective due to the inherent simplicity and elegance of the conformable fractional derivative definition.

The conformable fractional derivative, introduced in recent years, provides a more intuitive and direct approach to fractional calculus. This innovative definition has opened up new avenues for research and applications in various fields.

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CFBVPs, in particular, have gained prominence due to their potential to model complex phenomena with greater accuracy and efficiency. Several notable studies have contributed to our understanding of CFBVPs. These include:

Batarfi, et al. (2017)[15]: This study delved into solving a class of conformable fractional boundary values problems, where each solution is unique, is realisable.

Dong et al. (2019)[16]: This research focused on a stability analysis for conformable fractional dynamical systems, providing valuable insights into the behavior of solutions.

Zhong et al. (2018)[17]: This study explored the numerical solutions of conformable fractional differential equations, developing efficient algorithms for computational analysis.

Zhou and Zhang (2019)[18] : This research investigated the controllability of conformable fractional control systems, addressing the problem of steering the system to a desired state.

Faouzi (2020)[19]: This study examined a applications for conformable fractional calculus in modeling viscoelastic materials, demonstrating its potential in engineering and materials science.

Ahmadkhanlu (2023) [20]: It investigated the existence as well as the distinctiveness of solutions for a particular class of adaptable differential equations with fractions with delay.

These studies, among others, have significantly advanced our understanding of CFBVPs and their applications, paving the way for further research and development in this exciting field.

Inspired by the research above with this article, this study examines the fractional boundary value problem.

$$\mathcal{T}_\zeta \varkappa(\iota) = \psi(\iota, \varkappa(\iota)), \quad 0 \le \iota \le A, \tag{1.1}$$

$$\varkappa'(0) = 0, \varpi\varkappa(0) + \varsigma\varkappa(A) = \gamma, \tag{1.2}$$

Where is the order-conformable fractional derivative $1 < \zeta < 2$, and ψ: $[0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$, is a function with features that will be discussed later.

2. PRELIMINARIES

To enhance reader comprehension, we introduce essential notation and lemmas employed with our subsequent proof.
Definition 2.1 [7]: Let $\zeta \in (0,1]$ the conformable derivative of $\varphi: \mathbb{R}^{\ge 0} \to \mathbb{R}$ of order ζ is formulated as

$$D^\zeta \varphi(\mathrm{r}) = \lim_{\epsilon\to 0} \frac{\varphi(\mathrm{r}+\epsilon \mathrm{r}^{1-\zeta})}{\epsilon}. \tag{2.1}$$

If $D^\zeta \varphi(\mathrm{r})$ exists on $(0, b)$ then $D^\zeta f(0) = \lim_{\mathrm{r}\to 0} D^\zeta \varphi(\mathrm{r})$.
From Definition 2.2 to Lemma 2.6, we set $\zeta \in (k, k+1], k \in \mathbb{N}$.
Definition 2.2 [7]: The conformable derivative of $\varphi: \mathbb{R}^{\ge 0} \to \mathbb{R}$ is formulated as

$$D^\zeta \varphi(\mathrm{r}) = D^\varsigma f^{(k)}(t),$$

where $\varsigma = \zeta - k$.
Definition 2.3 [7]: The conformable integral of $\varphi: \mathbb{R}^{\ge 0} \to \mathbb{R}$ of order ζ is given by

$$I^\zeta \varphi(\mathrm{r}) = \frac{1}{k!} \int_0^{\mathrm{r}} (\mathrm{r} - \jmath)^k \jmath^{\zeta-k-1} \varphi(\jmath) d\jmath.$$

Lemma 2.4 [7]: For each $r > 0, D^\zeta I^\zeta \varphi(r) = \varphi(r)$ whenever φ is continuous on $\mathbb{R}^{\ge 0}$.
Lemma 2.5 [7] For all $r \in [0,1]$, $D^\zeta r^\ell = 0$ if $\ell = 1, \dots, k$.
Lemma 2.6 [7, 6] If $D^\zeta \varphi(r)$ is continuous on $\mathbb{R}^{\ge 0}$, then

$$I^\zeta D^\zeta \varphi(\mathrm{r}) = \varphi(\mathrm{r}) + C_1 + C_2 \mathrm{r}^2 + \cdots + C_n \mathrm{r}^n, \text{ for some real numbers } C.$$

3. MAIN RESULTS

In this section, we will present, rigorously demonstrate our primary findings. To lay the groundwork for these results, we will first establish several essential lemmas.

Lemma 3.1 Assume $1 < \zeta \le 2$ and $\varrho \in C([0, A])$. Then the solution of is

$$\begin{aligned} &\mathcal{T}_\zeta \varkappa(\iota) = \varrho(\iota), \\ &\varkappa'(0) = 0, \varpi\varkappa(0) + \varsigma\varkappa(A) = \gamma, \end{aligned} \tag{3.1}$$

$$\varkappa(\iota) = \int_0^\iota (\iota - \jmath)\jmath^{\zeta-2} \varrho(\jmath) d\jmath - \frac{\varsigma}{\varpi+\varsigma} \int_0^A (A-\jmath)\jmath^{\zeta-2} \varrho(\jmath) d\jmath + \frac{\gamma}{\varpi+\varsigma} \tag{3.2}$$

Proof. To establish our results, we utilize Lemma 2.6 to transform a boundary value problem (3.1) into an equivalent integral equation.

$$\varkappa(\iota) = \int_0^\iota (\iota - \jmath)\jmath^{\zeta-2} d\jmath + \gamma_1 + \gamma_2 \iota. \tag{3.3}$$

By differentitating from relation we have

$$\zeta'(\iota) = \int_0^{\iota} \jmath^{\zeta-2}\varrho(\jmath)d\jmath + \gamma_2,$$

From the first boundary condition we have

$$\varkappa'(0) = \gamma_2 = 0,$$

also

$$\varkappa(A) = \int_0^{\iota} (A-\jmath)\jmath^{\zeta-2}\varrho(\jmath)d\jmath + \gamma_1,$$

so from the second boundary condition we have

$$\varpi\varkappa(0) + \varsigma\varkappa(A) = \varpi\gamma_1 + \varsigma\int_0^{A} (A-\jmath)\jmath^{\zeta-2}\varrho(\jmath) + \gamma_1 = \gamma,$$

hence

$$(\varpi+1)\gamma_1 + \varsigma\int_0^{A} (A-\jmath)\jmath^{\zeta-2}\varrho(\jmath)d\jmath = \gamma,$$

so

$$\gamma_1 = \frac{-\varsigma}{\varpi+\varsigma}\int_0^{A} (A-\jmath)\jmath^{\zeta-2}\varrho(\jmath)d\jmath + \frac{\gamma}{\varpi+\varsigma}.$$

Consequently

$$\varkappa(\iota) = \int_0^{\iota} (\iota-\jmath)\jmath^{\zeta-2}\varrho(\jmath)d\jmath - \frac{\varsigma}{\varpi+\varsigma}\int_0^{A} (A-\jmath)\jmath^{\zeta-2}\varrho(\jmath)d\jmath + \frac{\gamma}{\varpi+\varsigma}. \tag{3.4}$$

Now we are ready to use some fixed-point theorems to prove the existence results. To establish our initial findings, we will employ the powerful Banach fixed point theorem. Before applying this theorem, we will introduce the following assumptions:

i For every $\varkappa, \xi \in \mathbb{R}$ and $\iota \in [0, A]$, there exists a constant $\Theta > 0$ such that

$$|\psi(\iota,\varkappa) - \psi(\iota,\xi)| \leq \Theta|\varkappa - \xi|.$$

ii The function $\psi \in C([0,A]\times\mathbb{R},\mathbb{R})$,

iii iii $\forall\ \iota \in [0,A],\ \ \forall\ \varkappa \in \mathbb{R},\ \ \exists\ \mathcal{K},\ |\psi(\iota,\varkappa)| \leq \mathcal{K}$

Theorem 3.2 Let the condition (i) hold and $\left(\frac{\Theta A^{\zeta}\left[1+\frac{|\varsigma|}{|\varpi+\varsigma|}\right]}{\zeta(\zeta-1)}\right) < 1$, then the conformable fractional boundary value problem (1.1)-(1.2) has only one solution.

Proof. At first, changing the conformable fractional boundary value problem (1.1)-(1.2) to a fixed point problem is necessary. We define the operator

$$\mathcal{G}(\varkappa)(\iota) = \int_0^{\iota} (\iota-\jmath)\jmath^{\zeta-2}\psi(\jmath,\varkappa(\jmath))d\jmath - \frac{\varsigma}{\varpi+\varsigma}\int_0^{A} (A-\jmath)\jmath^{\zeta-2}\psi(\jmath,\varkappa(\jmath))d\jmath + \frac{\gamma}{\varpi+\varsigma}. \tag{3.5}$$

To establish the existence of a unique solution for the conformable fractional boundary value problem (1.1)-(1.2), we will employ the powerful Banach contraction principle. It is evident that the fixed points of the operator $\mathcal{G}$ correspond directly to the solutions of this problem. Therefore, by demonstrating that $\mathcal{G}$ is a contraction mapping, we can guarantee the existence of a unique solution.

Assume $\varkappa, \xi \in C([0,A],\mathbb{R})$ and $\iota \in [0,A]$ be an arbitrarry quantity, then

$$\begin{aligned}
|\mathcal{G}(\varkappa)(\iota) - \mathcal{G}(\xi)(\iota)| &\leq \int_0^{\iota} (\iota-\jmath)\jmath^{\zeta-2}|\varkappa(\jmath) - \xi(\jmath)|d\jmath \\
&+ \frac{|\varsigma|}{|\varpi+\varsigma|}\int_0^{A} (A-\jmath)\jmath^{\zeta-2}|\varkappa(\jmath) - \xi(\jmath)|d\jmath \\
&\leq \Theta \parallel \varkappa - \xi \parallel \int_0^{\iota} (\iota-\jmath)\jmath^{\zeta-2}d\jmath \\
&+ \frac{|\varsigma|\Theta\|\varkappa-\xi\|}{|\varpi+\varsigma|}\int_0^{\iota} (A-\jmath)\jmath^{\zeta-2}d\jmath \\
&\leq \left(\frac{\Theta A^{\zeta}\left[1+\frac{|\varsigma|}{|\varpi+\varsigma|}\right]}{\zeta(\zeta-1)}\right) \parallel \varkappa - \xi \parallel_{\infty}.
\end{aligned}$$

So

$$\parallel \mathcal{G}(\varkappa) - \mathcal{G}(\xi) \parallel \leq \left(\frac{\Theta A^{\zeta}\left[1+\frac{|\varsigma|}{|\varpi+\varsigma|}\right]}{\zeta(\zeta-1)}\right) \parallel \varkappa - \xi \parallel_{\infty}.$$

Thus because of the assumption of the theorem, the operator $\mathcal{G}$ is a contraction mapping. By leveraging the powerful Banach fixed point theorem, we can confidently assert that the operator $\mathcal{G}$ possesses a unique fixed point. This, in turn, guarantees a unique solution to the conformable fractional boundary value problem we are investigating.

Now for the second result we will use Schaefer's fixed point theorem [21]. Let us to intorduce the following Lemmas.

Lemma 3.3 Suppose $(ii) - (iii)$ hold, then $\mathcal{G}$ is a continuous operator.

Proof. Assume $\varkappa_n$ be a sequence with the property $\varkappa_n \varkappa$ in $C([0,A],\mathbb{R})$ and $\iota \in [0,A]$

$$\begin{aligned}
|\mathcal{G}(\varkappa_n)(\iota) - \mathcal{G}(\varkappa)(\iota)| &\leq \int_0^{\iota} (\iota-\jmath)\jmath^{\zeta-2}|\psi(\varkappa_n(\jmath)) - \psi(\varkappa(\jmath))|d\jmath \\
&+ \frac{|\varsigma|}{|\varpi+\varsigma|}\int_0^{A} (A-\jmath)\jmath^{\zeta-2}|\psi(\varkappa_n(\jmath)) - \psi(\varkappa(\jmath))|d\jmath \\
&\leq \parallel \psi(.,\varkappa_n(.)) - \psi(.,\varkappa(.)) \parallel \int_0^{\iota} (\iota-\jmath)\jmath^{\zeta-2}d\jmath
\end{aligned}$$

$$+\frac{|\varsigma|\|\psi(.,\varkappa_n(.))-\psi(.,\varkappa(.))\|}{|\varpi+\varsigma|}\int_0^{\iota}(A-\jmath)\jmath^{\zeta-2}d\jmath$$
$$\leq\left(\frac{A^{\zeta}\left[1+\frac{|\varsigma|}{|\varpi+\varsigma|}\right]}{\zeta(\zeta-1)}\right)\|\psi(.,\varkappa_n(.))-\psi(.,\varkappa(.))\|_{\infty}.$$

From the continuity of the function ψ, we conclude

$$\|\mathcal{G}(\varkappa_n)-\mathcal{G}(\varkappa)\|\leq\left(\frac{\Theta A^{\zeta}\left[1+\frac{|\varsigma|}{|\varpi+\varsigma|}\right]}{\zeta(\zeta-1)}\right)\|\psi(.,\varkappa_n(.))-\psi(.,\varkappa(.))\|_{\infty}.$$

Since $\|\psi(.,\varkappa_n(.))-\psi(.,\varkappa(.))\|_{\infty}$. tends to 0 as $n\to infty$,. So $\|\mathcal{G}(\varkappa_n)-\mathcal{G}(\varkappa)\|_{\infty}=0$ and this complete the proof.

Lemma 3.4 Suppose $(ii)-(iii)$ hold, then $\mathcal{G}$ mapps bounded sets into bounded sets in $C([0,A],\mathbb{R})$.

Proof. Assume $\rho^*>0$ be a constant then $\mathcal{B}_{\rho^*}=\{\varkappa\in C([0,A],\mathbb{R})|\ \|\varkappa|_{\infty}\leq\rho^*\}$, let $\varkappa\in\mathcal{B}_{\rho^*}$. In view of (iii) for all $\iota\in[0,A]$ we have

$$|\mathcal{G}(\varkappa)(\iota)|\leq\int_0^{\iota}(\iota-\jmath)\jmath^{\zeta-2}|\psi(\jmath,\varkappa(\jmath))d\jmath$$
$$+\frac{|\varsigma|}{|\varpi+\varsigma|}\int_0^{A}(A-\jmath)\jmath^{\zeta-2}|\psi(\jmath,\varkappa(\jmath))d\jmath+\frac{|\gamma|}{|\varpi+\varsigma|}$$
$$\leq\mathcal{K}\int_0^{\iota}(\iota-\jmath)\jmath^{\zeta-2}|d\jmath$$
$$+\frac{|\varsigma|\mathcal{K}}{|\varpi+\varsigma|}\int_0^{A}(A-\jmath)\jmath^{\zeta-2}|d\jmath+\frac{|\gamma|}{|\varpi+\varsigma|}$$
$$\leq\frac{\mathcal{K}A^{\zeta}}{\zeta(\zeta-1)}+\frac{\mathcal{K}A^{\zeta}|\varsigma|}{\zeta(\zeta-1)|\varpi+\varsigma|}+\frac{|\gamma|}{|\varpi+\varsigma|}:=\eta$$

That is $\|\mathcal{G}(\varkappa)\|\leq\eta$ and this completes the proof.

Lemma 3.5 Suppose $(ii)-(iii)$ hold, then $\mathcal{G}$ mapps bounded sets into equicontinuous sets in $C([0,A],\mathbb{R})$.

Proof. Assume $\iota_1,\iota_2\in(0,A]$, such that $\iota_1<\iota_2$ and $\mathcal{B}_{\rho^*}\subset C([0,A],\mathbb{R})$ be a bounded set. Let $\varkappa\in\mathcal{B}_{\rho^*}$. We have

$$|\mathcal{G}(\varkappa)(\iota_2)-\mathcal{G}(\varkappa)(\iota_1)|\leq\left|\int_0^{\iota_1}[\iota_2-\iota_1]\jmath^{\zeta-2}\psi(\jmath,\varkappa(\jmath))d\jmath+\int_{\iota_1}^{\iota_2}(\iota_2-\jmath)\jmath^{\zeta-2}\psi(\jmath,\varkappa(\jmath))d\jmath\right|$$
$$\leq\frac{\mathcal{K}\iota_1^{\zeta-1}}{\zeta-1}(\iota_2-\iota_1)+\frac{\mathcal{K}}{\zeta(\zeta-1)}[\iota_2^{\zeta}-\iota_1^{\zeta}].$$

Because of the above relation, we see that the right-hand side of the relation tends to zero if ι_1 tends ι_2. That is $\mathcal{G}$ mapps bounded sets into equicontinuous sets in $C([0,A],\mathbb{R})$.

Theorem 3.6 Suppose $(ii)-(iii)$ hold, then conformable fractional boundary value problem (1.1)-(1.2) has at least one solution on $[0,A]$.

Proof.

Consider

$$\Delta=\{\varkappa\in C([0,A],\mathbb{R})|\varkappa=\delta\mathcal{G}(\varkappa);\quad 0<\delta<1\}.$$

We claim that Δ is bounded. Assume $\varkappa\in\Delta$, then there exist $0<\delta<1$ such that $\varkappa=\delta\mathcal{G}$. So for all $\iota\in[0,A]$ we have

$$\varkappa(\iota)=\delta\big(\int_0^{\iota}(\iota-\jmath)\jmath^{\zeta-2}\psi(\jmath,\varkappa(\jmath))d\jmath$$
$$-\frac{\varsigma}{\varpi+\varsigma}\int_0^{A}(A-\jmath)\jmath^{\zeta-2}\psi(\jmath,\varkappa(\jmath))d\jmath+\frac{\gamma}{\varpi+\varsigma}\Big)$$

Now by (iii) for every $\iota\in[0,A]$ we get

$$|\mathcal{G}|\leq\int_0^{\iota}(\iota-\jmath)\jmath^{\zeta-2}|\psi(\jmath,\varkappa(\jmath))|d\jmath$$
$$+\frac{|\varsigma|}{|\varpi+\varsigma|}\int_0^{A}(A-\jmath)\psi(\jmath,\varkappa(\jmath))d\jmath+\frac{|\gamma|}{|\varpi+\varsigma|}$$
$$\leq\mathcal{K}\int_0^{\iota}(\iota-\jmath)\jmath^{\zeta-2}d\jmath+\frac{|\varsigma|\mathcal{K}}{|\varpi+\varsigma|}\int_0^{A}(A-\jmath)d\jmath+\frac{|\gamma|}{|\varpi+\varsigma|}$$
$$\leq\frac{\mathcal{K}A^{\zeta}}{\zeta(\zeta-1)}+\frac{\mathcal{K}A^{\zeta}|\varsigma|}{\zeta(\zeta-1)|\varpi+\varsigma|}+\frac{|\gamma|}{|\varpi+\varsigma|}.$$

The relation above states that the set Δ is bounded. This is a crucial assumption for applying Schaefer's fixed point theorem. The theorem states that if an operator is completely continuous and has a fixed point property (meaning there exists a point $\varkappa$ such that $\mathcal{G}(\varkappa)=\varkappa$), then it has at least one fixed point [21].

In this case, the operator $\mathcal{G}$ is completely continuous given Lemmas 3.3-3.5, and maps bounded sets into bounded sets. Therefore, by Schaefer's fixed point theorem, we can conclude that $\mathcal{G}$ has at least one fixed point. This fixed point corresponds to a solution of the conformable fractional boundary value problem (1.1)-(1.2). Therefore, the existence of at least one solution for this problem is guaranteed.

4. EXAMPLE

To underscore the efficacy of our findings, here is an example to help illustrate: Reflect on the fractional boundary value problem:

$$\mathcal{T}_{\zeta}\varkappa(\iota) = \psi(\iota, \varkappa(\iota)), \iota \in [0,1], \zeta \in (1,2)$$
$$\varkappa(0) + \varkappa(1) = 0, \varkappa'(0) = 0, \tag{4.1}$$

where $\psi(\iota, \varkappa(\iota)) = \frac{|\varkappa(\iota)|(\sin(\iota)+2e^{-\iota})}{(5+|\sin\iota|(1+|\varkappa(\iota)|)}$. This example showcases the applicability and effectiveness of our proposed methods in addressing a concrete problem within the realm of conformable fractional calculus.

$$|\psi(\iota,\varkappa) - \psi(\iota,\xi)| \leq \frac{\sin\iota + 2e^{-\iota}}{5+|\sin\iota|}\left|\frac{\varkappa}{1+\varkappa} - \frac{\xi}{1+\xi}\right|$$
$$\leq \frac{\sin\iota + 2e^{-\iota}}{5+|\sin\iota|}\frac{|\varkappa-\xi|}{(1+\varkappa)(1+\xi)}$$
$$\leq \frac{\sin\iota + 2e^{-\iota}}{5+|\sin\iota|}|\varkappa - \xi|$$
$$\leq \frac{1}{10}|\varkappa - \xi|.$$

So the assumption (i) is satisfied with $\Theta = \frac{1}{5}$. Now we check the hypothesis of the theorem 3.2. In this problem $\varpi = \varsigma = A = 1$, hence for $\zeta = \frac{3}{2}$,

$$\left(\frac{\Theta A^{\zeta}\left[1+\frac{|\varsigma|}{|\varpi+\varsigma|}\right]}{\zeta(\zeta-1)}\right) = \frac{\frac{1}{5}\left[1+\frac{1}{2}\right]}{\frac{31}{22}} = \frac{2}{5}.$$

Consequently all conditions of the theorem 3.2 are hold and the conformable fractional boundary value problem (4.1) has a unique solution.

5. CONCLOSION

In this work, we studied a class of conformable fractional boundary value problems with two-point boundary conditions. Using the Banach contraction mapping principle and Shaefer's fixed point theorem, some necessary and sufficient conditions imposed the right-hand side function of the conformable fractional equation to guarantee the problem's solution. Two examples were presented to illustrate the efficiency of the main results. This work has shown that a class of fractional boundary value problems under some conditions can have a unique solution, such as a natural order boundary value problem.

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CONFLICTS OF INTEREST

The authors declare no conflict of interest

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